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Kinetic theory of ion acoustic waves in a plasma with collisional electrons

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An analytical theory of ion acoustic waves in Maxwellian plasmas and in plasmas with an externally applied temperature gradient has been developed. The emphasis is upon the effects of electron-electron and electron-ion collisions on ion acoustic wave damping and on the effective electron heat conductivity associated with the waves. The asymptotic limits of weakly and strongly collisional electrons are studied analytically and then a numerical solution is compared to Fokker-Planck simulations and fitted by simple algebraic expressions. The limit of large Z has also been investigated and the condition of validity for the Lorentzian plasma approximation has been found. It is also shown that the heat flux driven ion acoustic instability exists over a wide range of parameters with a smooth transition from the collisionless to the collisional limit. Accounting for ion-ion collisions leads to the separation of the instability region in two domains. The long-wavelength part dominates in plasmas with a relatively high ion temperature.

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I. INTRODUCTION

In laser-plasma interaction physics it is common to encounter situations where the effects of particle collisions can have an important impact on wave and transport processes. By the use of short-wavelength lasers and correspondingly higher plasma densities the characteristic scale of parametrically excited plasma waves can become comparable with the ion and electron mean free paths where the standard plasma descriptions of weakly or strongly collisional cases are no longer valid. To deal with these conditions several practical quasihydrodynamical models have recently been proposed. These are based on the idea of delocalization of the transport coefficients in plasmas over the scale of the electron mean free path [1–5]. The performance of these models is strongly reliant upon exact solutions of the kinetic Fokker-Planck equations, which are difficult to obtain analytically in this intermediate region and therefore the implementation of complicated numerical codes is involved. The development of a correct analytical description of this region is an important challenge for theoretical plasma physics because of its relevance to basic problems from kinetic theory and many applications.

Laser absorption in plasmas is accompanied by strong heat fluxes that can be the source of secondary plasma instabilities. One of them, the return current instability, has been discovered by Forslund [6] and is involved in many processes associated with laser-plasma interaction like laser energy absorption, heat transport, and stimulated scattering. This instability is induced by the return current of cold electrons which move in order to compensate for the charge separation produced by hot, heat flux carrying electrons. Although originally discovered in the limit of collisionless electrons, it has recently been found in the collisional case as well [7]. Using a systematic kinetic approach we confirm here the existence of the hydrodynamical heat flux instability and show that from the kinetic point of view both these instabilities have a common origin and merge smoothly in the intermediate region of wavelengths. We will also study the effect of ion-ion collisions on the instability.

We have obtained ion acoustic wave dispersion characteristics in a wide parameter range. The electron part contains contributions from both electron-electron and electron-ion collisions. Electron-ion collisions are described using the procedure of summation of angular harmonics of the electron distribution function in the same way as in Refs. [2,8]. This provides a smooth transition into the collisionless regime. In the limit of short wavelengths electron-electron collisions are described by a technique similar to the asymptotic expansion that has recently been proposed by Maximov and Silin [9–11]. In the other limit of long-wavelengths the ion acoustic wave dispersion is found by using perturbation theory about the hydrodynamic solution. The ion contribution is described by the generalized Grad moment expansion, which gives the correct response for an arbitrary ratio of the ion acoustic frequency to the ion-ion collision rate [7]. It is shown that heat flux driven ion acoustic instability can be excited in a wide spectral range and the threshold of the collisional instability is even lower than the known collisionless threshold.

Analytical expressions for the electron contribution to ion acoustic wave damping in Maxwellian plasmas have been derived. A short account of these results has already been published in Ref. [12] where the asymptotic expressions that are valid in the limit of short wavelengths have been found. Similar expressions for the ion

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acoustic damping have also been reported in Ref. [13]. Intermediate asymptotics exists in between the purely collisional and collisionless limits and originate from the electron-electron contribution to the electron distribution function in the region of small electron velocities. The contribution from electron-electron collisions to the ion acoustic wave damping resolves the long contradictory discussion on the subject starting from the paper of Kulsrud and Shen [14] and continuing until now [8,15,16]. Our results clearly show that the Lorentz plasma approximation [8,15] (omitting the electron-electron collisions in the limit of $Z \to \infty$) fails to work for any practical ionization ion state. In high-Z plasmas even small amounts of electron-electron collisions have a significant effect on the ion acoustic damping and especially on the electron heat conductivity. This result has been obtained recently by Epperlein [16] by numerical Fokker-Planck simulations. We now provide an analytical explanation of these findings and also explain the problem dependency of the electron heat conductivity coefficient [17]. We emphasize the importance of correctly accounting for the electron density and temperature perturbations.

Our paper is organized in the following way. In Sec. II we derive the linearized kinetic description of the electron response for ion acoustic waves in plasmas with an externally applied temperature gradient. Section III deals with the solution to the equation for the symmetric part of the electron distribution function which covers the wide range of collisionality from the collisionless into the electron-electron collision dominated region. In Sec. IV we analyze the expressions for the ion acoustic damping and electron heat conductivity in Maxwellian plasmas and make a comparison with Fokker-Planck numerical calculations. By virtue of two asymptotic analytical solutions we construct simple expressions that describe ion acoustic damping and electron heat conductivity in the entire range of wave numbers. In Sec. V we derive expressions for the growth rate of the return current instability and discuss the effects of the electron and ion collisions on its performance. It is shown that the ion contribution to the ion acoustic wave damping separates the short- and long-wavelength parts of the return current instability. It increases the threshold of short-wave excitation, but has almost no effect on the long-wavelength instability. Section VI summarizes our results and discusses their possible applications.

II. KINETIC DESCRIPTION OF THE ELECTRON RESPONSE TO ION ACOUSTIC WAVES

A. The initial state of the plasma with an externally supported heat flux

In our studies of ion acoustic wave dispersion and damping we consider as a reference state a plasma with a temperature gradient supported by some unspecified external source. The spatial scale of the temperature inhomogeneity along the z axis, L_T, is assumed to be sufficiently large so that the classical collisional description can be applied to describe this reference state. The ions will be treated as a cold liquid throughout the paper with the exception of the last part of Sec. IV and in Sec. V where the ion contribution to the ion acoustic damping will be accounted for in terms of the Grad 21-moment approximation as derived in our previous paper [7].

With these assumptions in mind our governing equations constitute a kinetic equation for the electron distribution function $f_e(\mathbf{v},z,t)$, which will be close to a local Maxwellian $F_0(v,z) = (n_0/v_{Te}^3)(2\pi)^{-3/2}\exp(-v^2/2v_{Te}^2)$ in the ion reference system, with a spatially inhomogeneous density $n_0(z)$ and temperature $T_{e0}(z) = m_e v_{Te}^2$. The ions are described by the continuity and Euler equations. We are interested in temperature inhomogeneity scales much larger than a Debye length, hence we can substitute the Poisson equation by the charge neutrality condition $n_e = Zn_i$. The electron kinetic equation reads

$$\frac{\partial f_e}{\partial t} + v_z\frac{\partial f_e}{\partial z} + \frac{e}{m_e}\frac{\partial \phi}{\partial z}\frac{\partial f_e}{\partial v_z} = C_{ei}[f_e] + C_{ee}[f_e,f_e], \quad (1)$$

where $-e$ and m_e stand for the electron charge and mass and $\phi(z,t)$ is the electric potential. The electron-ion collision term

$$C_{ei}[f_e(\mathbf{v})] = \frac{1}{2}\nu_{ei}(v)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f_e}{\partial \mu} \quad (2)$$

is written neglecting electron-ion energy exchange, where $\mu = v_z/v$ is the cosine of the angle between the electron velocity and the z axis, $\nu_{ei}(v) = 4\pi Z n_0 e^4\Lambda_e/m_e^2 v^3$ is the velocity-dependent collision rate, and Λ_e is the Coulomb logarithm.

The expression for the electron-electron collision term C_{ee} is more complicated because it is nonlinear with respect to f_e and contains integral terms [18]. In highly ionized plasmas, $Z \gg 1$, C_{ee} is Z times smaller than the electron-ion collision term but is still important because it is responsible for energy redistribution between the electrons. For this reason we will only account for electron-electron collisions in the equation for the symmetric part of the electron distribution function because here the electron-ion collision term makes a negligible contribution. It is known that this kind of approximation is only valid for very large Z, but following Epperlein [16] we will later redefine the electron mean free path in such a way that our expressions will be good for small Z as well. With all these assumptions the electron-electron collision term can be written as [16,18]

$$C_{ee}[f_1(v),f_2(v)] = \frac{4\pi}{n_0 Z}\nu_{ei}(v)v\frac{\partial}{\partial v}\left[f_2(v)\int_0^v f_1(w)w^2dw + \frac{v}{3}\frac{\partial f_2}{\partial v}\left(v^{-2}\int_0^v f_1(w)w^4dw + v\int_v^\infty f_1(w)w\,dw\right)\right]. \quad (3)$$

The ion density n_i and ion velocity $u_i = u_{iz}$ are governed by the continuity and Euler equations

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z}(n_i u_i) = 0, \qquad \frac{\partial u_i}{\partial t} = -Z\frac{e}{m_i}\frac{\partial \phi}{\partial z} + \frac{1}{n_i m_i}R_{ie}, \tag{4}$$

with the friction force $R_{ie} = m_e \int d\mathbf{v} v_z \nu_{ei}(v) f_e(\mathbf{v})$. Emphasizing the effects of electron collisionality we will assume that ions are cold and collisionless. In Secs. IV and V we will account for ion dissipation of the ion acoustic wave by using the results of the generalized Grad moment expansion derived in our previous paper [7].

As a reference state we consider a stationary plasma ($\partial/\partial t = 0$) without hydrodynamical motion ($u_i = 0$) and with a weakly inhomogeneous temperature $L_T \gtrsim 100\lambda_{ei}$ such that nonlocal transport effects are negligible. Here $\lambda_{ei} = 3\sqrt{\pi/2}v_{Te}/\nu_{ei}(v_{Te})$ is the electron mean free path. The electron distribution function consists of a Maxwellian part F_0 plus a small anisotropic correction that is proportional to λ_{ei}/L_T: $f_e(\mathbf{v}) = F_0(1 + \mu\psi_\nabla)$. Substituting into Eq. (1) gives us an expression for ψ_∇,

$$\psi_\nabla(v) = \frac{v}{\nu_{ei}(v)}\left(\frac{e}{T_{e0}}\frac{d\phi_0}{dz} - \frac{d}{dz}\ln F_0\right), \tag{5}$$

which comes from the electron-ion collision term. The contribution to the isotropic part of distribution function from the electron-electron collision term can be neglected here because it is second order in the expansion parameter λ_{ei}/L_T. The ion Euler equation (4) and condition of the absence of electric current $\int dv v^3 \psi_\nabla F_0 = 0$ provides us with relations between spatial gradients of the hydrodynamical quantities:

$$n_0 T_{e0} = \text{const}, \qquad e\frac{d\phi_0}{dz} = \frac{3}{2}\frac{dT_{e0}}{dz}. \tag{6}$$

If now we specify the definition of temperature scale length as $1/L_T = d\ln T_{e0}/dz$ then we have the following expression for the anisotropic part of the electron distribution function:

$$\psi_\nabla(v) = \frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{\lambda_{ei}}{L_T}\frac{v^4}{v_{Te}^4}\left(4 - \frac{v^2}{2v_{Te}^2}\right). \tag{7}$$

Equations (6) and (7) provide the full definition of the reference state whose stability will be investigated in the following sections.

B. Linearized electron kinetic equation

We consider the propagation of a small-amplitude ion acoustic wave with frequency ω and wave number $k \gg 1/L_T$ in the above-described plasma. As is already known from previous papers [6,7], the most unstable waves propagate in the direction of the temperature gradient. Because of this we restrict ourselves to the one-dimensional case and assume that the disturbances of the electric potential $\delta\phi$, electron distribution function δf_e, ion density δn_i, and velocity u_i have an exponential spatiotemporal dependence $\propto \exp(-i\omega t + ikz)$. The disturbance of the electron distribution function is expanded in a series of Legendre polynomials $P_l(\mu)$ which are eigenfunctions of the electron-ion collision operator, $\delta f_e(v,\mu) = \sum_{l=0}^{\infty} \delta f_l(v) P_l(\mu)$. We are interested in low-frequency phenomena $\omega \sim kc_s \ll kv_{Te}$ where $c_s = (ZT_{e0}/m_i)^{1/2}$ and so the time derivative term in Eq. (1) is small and we keep it only to the lowest order in the equation for the symmetric part of the distribution function δf_0. Formally the stationary approximation for angular harmonics with $l \geq 1$ assumes that $\omega \ll \nu_{ei}$ but in fact it is also valid in the opposite limit as far as the wave phase velocity is small, $\omega/k \ll v_{Te}$. We also neglect the stationary potential ϕ_0 (6) in the equations for δf_l, which has a much smaller effect than that of the anisotropic part (7) of the reference distribution function itself. Finally, we have to recall that the collisional operators are defined in the ion reference frame. Therefore the ion velocity u_i will show up explicitly in the equations for δf_0 and δf_2. The full set of equations for the angular harmonics can be written as follows:

$$-i\omega\delta f_0 + \frac{i}{3}kv\delta f_1 + \frac{i}{3}\frac{e}{m_e}k\delta\phi\left(\frac{\partial}{\partial v} + \frac{2}{v}\right)\psi_\nabla F_0 - \frac{i}{3}kvu_i\frac{\partial F_0}{\partial v} = C_{ee}[\delta f_0], \qquad (l=0) \tag{8}$$

$$ikv\delta f_0 + \frac{2}{5}ikv\delta f_2 + i\frac{e}{m_e}k\delta\phi\frac{\partial F_0}{\partial v} = -\nu_{ei}\delta f_1 \qquad (l=1), \tag{9}$$

$$\frac{2}{3}ikv\delta f_1 + \frac{3}{7}ikv\delta f_3 + i\frac{2}{3}\frac{e}{m_e}k\delta\phi\left(\frac{\partial}{\partial v} - \frac{1}{v}\right)\psi_\nabla F_0 - \frac{2}{3}ikvu_i\frac{\partial F_0}{\partial v} = -3\nu_{ei}\delta f_2 \quad (l=2), \tag{10}$$

$$\frac{l}{2l-1}ikv\delta f_{l-1} + \frac{l+1}{2l+3}ikv\delta f_{l+1} = -\frac{1}{2}l\,(l+1)\nu_{ei}\delta f_l \qquad (l>2), \tag{11}$$

where $C_{ee}[\delta f_0] \equiv C_{ee}[F_0, \delta f_0] + C_{ee}[\delta f_0, F_0]$ is the linearized isotropic part of electron-electron collisional operator.

The common strategy for solving of this infinite set of coupled equations consists of the assumption that the higher-order angular harmonics are small and a reasonable approximation will be achieved if only two of them are kept (cf., for example, Refs. [9,11]). This is indeed the case for the collisional region but this approximation fails for collisionless electrons. In the case of collisionless electrons the main contribution to the Landau damping comes from electrons propagating almost across the wave vector and this corresponds to high-l harmonics. The procedure for the summation of all angular harmonics has been described in Refs. [2,8,19]. The main idea consists of solving Eq. (11). If we put the second term on the left-hand side of this equation into its right-hand side and introduce the modified collision frequency,

$$\tilde{\nu}_l = \frac{1}{2} l\,(l+1)\nu_{ei} + ikv\frac{l+1}{2l+3}\frac{\delta f_{l+1}}{\delta f_l}, \tag{12}$$

then the formal solution of Eq. (11) reads

$$\delta f_l = -i\frac{l}{2l-1}\frac{kv}{\tilde{\nu}_l}\delta f_{l-1}. \tag{13}$$

If we substitute this solution (13) back into Eq. (12), then a recurrent formula for $\tilde{\nu}_l$ appears,

$$\tilde{\nu}_{l-1} = \frac{1}{2} l\,(l-1)\nu_{ei} + \frac{l^2}{4l^2-1}\frac{k^2v^2}{\tilde{\nu}_l}, \tag{14}$$

which completes the formal solution of Eq. (11). In fact, it is enough to calculate $\tilde{\nu}_1 = \nu_{ei} H_1(kv/\nu_{ei})$, because all necessary functions can be expressed through it explicitly. The function H_1 can be written as a continued fraction, but in Ref. [8] the simple approximation $H_1(x) = [1+(\pi x/6)^2]^{1/2}$ was proposed, which has the proper asymptotics and deviates from exact solution by less than 10% when $x \sim 1$.

Substitution of Eq. (13) for $l=3$ into Eq. (10) provides us with an expression for the second harmonic δf_2. Substituting this into Eq. (9) provides the first harmonic equation

$$\delta f_1 = -i\frac{kv}{\tilde{\nu}_1}\left(\delta f_0 - \frac{e\delta\phi}{T_{e0}}F_0\right) - \frac{\tilde{\nu}_1-\nu_{ei}}{\tilde{\nu}_1}\frac{u_i v}{v_{Te}^2}F_0 - \frac{\tilde{\nu}_1-\nu_{ei}}{\tilde{\nu}_1}\frac{e\delta\phi}{m_e}\frac{\partial}{\partial v}\frac{\psi_\nabla F_0}{v}, \tag{15}$$

which in turn allows us to obtain an equation for the symmetric part of the electron distribution function

$$\left(\frac{k^2v^2}{3\tilde{\nu}_1} - i\omega\right)\left(\delta f_0 - \frac{e\delta\phi}{T_{e0}}F_0\right) = i\omega\frac{e\delta\phi}{T_{e0}}F_0 - \frac{v^2}{3v_{Te}^2}iku_i H_1^{-1}F_0 - i\frac{e}{m_e v}k\delta\phi\left(\psi_\nabla F_0 + \frac{v^2}{3H_1}\frac{\partial}{\partial v}\frac{\psi_\nabla F_0}{v}\right) + C_{ee}[\delta f_0]. \tag{16}$$

From the last two equations it is evident that the symmetric part of the distribution function can be written as

$$\delta f_0 = \frac{e\delta\phi}{T_{e0}}[1+\psi_0(v)]F_0 \tag{17}$$

and Eq. (16) can be written for ψ_0, because $C_{ee}[\delta f_0]$ is a linear operator and the Maxwellian distribution function is a collisional invariant.

This equation is not closed because it contains the ion velocity that depends on the electric potential and on moments of the electron distribution function according to Eqs. (4). Making use of definitions (15) and (17) we can evaluate the friction force in terms of ψ_0 and write relations between the ion quantities and electric potential as follows:

$$\omega\frac{\delta n_i}{n_i} = ku_i, \qquad \omega u_i = kc_s^2\frac{e\delta\phi}{T_{e0}}(1+J_R), \tag{18}$$

where J_R is the integral distribution function related to the friction. It is convenient to introduce the new function

$$\psi_1 = \psi_0 - i\frac{\omega\nu_{ei}}{k^2v_{Te}^2}(H_1-1)(1+J_N) - i\frac{\nu_{ei}v_{Te}^2}{kvF_0}(H_1-1)\frac{d}{dv}\frac{\psi_\nabla F_0}{v} \tag{19}$$

and then define the following integrals

$$J_N = \frac{4\pi}{n_0}\int_0^\infty dv v^2\psi_0 F_0,$$
$$J_R = \frac{4\pi}{3n_0v_{Te}^2}\int_0^\infty dv\frac{v^4}{H_1}\psi_1 F_0, \tag{20}$$

where J_N, according to Eq. (17), is related to the electron density perturbation $\delta n_e/n_e = (4\pi/n_0)\int_0^\infty dv v^2\delta f_0 = (1+J_N)e\delta\phi/T_{e0}$. Equations (18) together with the quasineutrality condition, $Z\delta n_i = \delta n_e$, give us the ion acoustic dispersion equation in terms of moments of the electron distribution function

$$\frac{\omega^2}{k^2c_s^2} = \frac{1+J_R}{1+J_N}. \tag{21}$$

Substitution of Eqs. (18) and (21) into (16) gives us a closed equation for ψ_0,

$$\left(\frac{k^2v^2}{3\tilde{\nu}_1} - i\omega\right)\psi_0 F_0 = i\omega\left(1 - \frac{v^2}{3v_{Te}^2}\frac{1+J_N}{H_1}\right)F_0 - ikv_{Te}^2\left(1+\frac{v}{3H_1}\frac{d}{dv}\right)\frac{\psi_\nabla F_0}{v} + C_{ee}[\psi_0 F_0]. \tag{22}$$

This equation accounts accurately for the effects of electron-electron and electron-ion collisions together with collisionless electron Landau damping. It coincides with the corresponding equation for the symmetric part of the electron distribution function in Ref. [8] when the electron-electron collision term is discarded. It also coincides with the corresponding equation in Ref. [11], if one neglects the contribution from higher angular harmonics by putting $H_1 = 1$. We shall see later that both effects are important. One of them provides the proper expression for the collisional damping and the other provides the proper transition into the collisionless limit. In the intermediate region their contributions are comparable.

Before discussing the solution to Eq. (22) we will compare its terms. First of all we recall that according to Eq. (21) $\omega \sim kc_s$. Now considering the contribution from thermal electrons $v \sim v_{Te}$ and comparing the two terms in the brackets on the left-hand side of Eq. (22) we see that the first of them dominates in the region

$$k\lambda_{ei} > c_s/v_{Te}, \tag{23}$$

where we have assumed that $\tilde{\nu}_1 \approx v_{Te}/\lambda_{ei}$. Since here we restrict ourselves to the analysis of the short-wavelength region (23), then only the first term in parentheses on the left-hand side of Eq. (22) needs to be kept. The first two terms on the right-hand side of this equation are comparable, if $\lambda_{ei}/L_T \sim c_s/v_{Te}$. As we will see later this ordering corresponds to the threshold of the ion acoustic instability. Finally, the electron-electron collision term is of the order of ν_{ei}/Z and will be small in comparison with the left-hand side if

$$k^2\lambda_{ei}^2 > 1/Z. \tag{24}$$

Here we again consider the contribution from thermal electrons, $v \sim v_{Te}$. For all realistic ionizations the last inequality is much more restrictive than (23). However, the electron-electron collision term should be kept even in the short-wavelength region (24) because it ensures the convergence of the distribution function in the domain of small velocities, $v \ll v_{Te}$, and for this reason can make a significant contribution to the electron density and temperature perturbations that will diverge otherwise [11,12].

III. ISOTROPIC PART OF THE ELECTRON DISTRIBUTION FUNCTION

A. Laguerre expansion

In the physically interesting region where the condition $k\lambda_{ei} > c_s/v_{Te}$ (23) is satisfied, comparison of the leading term on the left-hand side of Eq. (22) with the first term on the right-hand side shows that $|\psi_0| \sim \omega/kv_{Te} \ll 1$. Therefore, J_N (20) is also much less than one and we can omit it in the right-hand side of Eq. (22). After that Eq. (22) can be written as

$$\left(\frac{k^2v^2}{3\nu_{ei}(v)}H_1^{-1} - i\omega\right)\psi_0 = i\omega\left[1 - \frac{v^2}{3v_{Te}^2}H_1^{-1}\right] - ikv_{Te}^2\left(1 - \frac{v^2}{3H_1v_{Te}^2} + \frac{v}{3H_1}\frac{d}{dv}\right)\frac{\psi_\nabla}{v} + F_0^{-1}C_{ee}[\psi_0F_0]. \tag{25}$$

Here we also give the explicit form of the collision operator

$$C_{ee}[\psi_0F_0] = \frac{2}{Z\sqrt{\pi}}\nu_{ei}(v)v\frac{d}{dv}\left\{F_0(v)G\left[\frac{d}{dv}\left(\frac{v_{Te}^2}{v}\frac{d\psi_0}{dv}\right)\right]\right\}, \tag{26}$$

where the functional $G[h(v)]$ reads

$$G[h] = \int_0^v dwh(w)\left[\gamma\left(\frac{3}{2},\frac{w^2}{2v_{Te}^2}\right) - \frac{w^3}{3\sqrt{2}v_{Te}^3}\exp\left(-\frac{w^2}{2v_{Te}^2}\right)\right] - \frac{v^3}{3\sqrt{2}v_{Te}^3}\int_v^\infty dwh(w)\exp\left(-\frac{w^2}{2v_{Te}^2}\right), \tag{27}$$

and $\gamma(3/2,x) = \int_0^x dx\sqrt{x}\exp(-x)$ is the generalized incomplete γ function [20].

Equation (25) is solved by first changing to energy units, $x = v^2/2v_{Te}^2$, and then expanding $\psi_0(x)$ in generalized Laguerre polynomials [20]

$$\psi_0(x) = i\frac{\omega}{kv_{Te}}\sum_{n=0}^{\infty}c_nL_n^{(1/2)}(x). \tag{28}$$

This expression once substituted into Eq. (25) gives

$$D(x)\sum_n c_nL_n^{(1/2)}(x) = S(x) + \frac{1}{kv_{Te}}\frac{1}{F_0(x)}C_{ee}\left[\sum_n c_nL_n^{(1/2)}(x)F_0(x)\right], \tag{29}$$

where

$$D(x)=\frac{8}{9\sqrt{\pi}}\frac{k\lambda_{ei}}{H_1}x^{5/2}-i\frac{c_s}{v_{Te}} \tag{30}$$

is responsible for the particle diffusion, $H_1 \equiv H_1[(4/3)\sqrt{2/\pi}k\lambda_{ei}x^2]$, and the source term S consists of the two parts $S(x)=S_T(x)+S_\nabla(x)$

$$\begin{aligned} S_T &= 1-\frac{2}{3}\frac{x}{H_1}, \\ S_\nabla &= -\frac{4}{3\sqrt{\pi}}\frac{v_{Te}}{c_s}\frac{\lambda_{ei}}{L_T}x^{3/2} \\ &\quad\times\left[4-x+\frac{2}{3H_1}\left(x^2-\frac{13}{2}x+6\right)\right]. \end{aligned} \tag{31}$$

The second part, $S_\nabla(x)$, is proportional to the temperature gradient.

Determination of the unknown expansion coefficients c_n constitutes the solution to the problem. An approximate solution can then be obtained by truncating the infinite sum into a finite one, $n \lesssim N$. Taking the inner product of Eq. (29) with the $L_m^{(1/2)}(x)$ gives rise to the following linear system of equations

$$\sum_{n=0}^{N} A_{mn}c_n = b_m. \tag{32}$$

Elements $A_{mn} = D_{mn} + C_{mn}$ contain the contribution from the diffusion term in the left-hand side of Eq. (29)

$$D_{mn}=\int_0^\infty dx\sqrt{x}e^{-x}L_m^{(1/2)}(x)L_n^{(1/2)}(x)D(x), \tag{33}$$

and from the electron-electron collision term

$$C_{mn}=\frac{3}{Zk\lambda_{ei}}\int_0^\infty dxe^{-x}L_{m-1}^{(3/2)}(x)L_{n-1}^{(3/2)}(x)\gamma(3/2,x)-\frac{2}{Zk\lambda_{ei}}\left[\frac{1}{m-1}\int_0^\infty dxe^{-2x}L_{n-1}^{(3/2)}(x)L_{m-2}^{(5/2)}(x)x^{5/2}+(m\rightleftharpoons n)\right]. \tag{34}$$

The right-hand side of Eq. (32),

$$b_m=\int_0^\infty dx\sqrt{x}e^{-x}L_m^{(1/2)}(x)S(x), \tag{35}$$

describes the contribution from the source term. Some of the details of the derivation of the matrix elements of the collision operator C_{mn} are given in Appendix A. The matrix elements were calculated and A_{mn} inverted by using the *Mathematica* program [21]. Examples of the numerical solution in the limits of small and large wavelengths are shown in Figs. 1, 2, and 3 for $Z/A=\frac{1}{2}$. They compare very well with the asymptotic solutions described in Secs. III B and III C in the limits of short and long wavelengths.

For values of the collisionality parameter $k\lambda_{ei} \lesssim 1$ an accurate solution can be obtained quickly using $N \lesssim 10$ polynomials. When $k\lambda_{ei} \gtrsim 1$, however, 30–50 polynomials are required. For larger values of $k\lambda_{ei}$ this becomes much worse and some alternative method must be sought. This is the motivation for the special analysis of the weakly collisional region in Sec. III C. We would also would like to obtain approximate analytical expressions for the ion acoustic wave damping and the electron heat conductivity in such a way that they will smoothly describe the transition between collisional and collisionless limits. Because the Laguerre expansion method described here is restricted by the condition $k\lambda_{ei} > c_s/v_{Te}$ and does not allow us to reach the strongly collisional region we should find an estimate of the characteristic wave number where the deviation from classical hydrodynamic expressions occurs. The classical hydrodynamic expressions are derived from Eq. (22) in Sec. III B.

B. Strongly collisional limit

The well-developed method of solution to the kinetic equation in the collisionally dominant region is based on the Chapman-Enskog method [22] and constitutes of a polynomial expansion of the function ψ_0 in powers of the electron energy. An expansion in the first two Laguerre polynomials recovers the classical hydrodynamic expressions from Eq. (22). The properties of particle and energy conservation for collisions, $\int_0^\infty C_{ee}v^2dv = 0$ and $\int_0^\infty C_{ee}v^4dv = 0$ described also by Eq. (A5) supplies us with two constraints on the electron distribution function which allow us to determine the unknown coefficients in the expansion. The condition of particle conservation reads

$$\int_0^\infty dxx^3H_1^{-1}\psi_1e^{-x}=0. \tag{36}$$

The energy conservation principle can be written in terms of the integral

$$J_Q=\frac{16}{3\sqrt{2\pi}}\int_0^\infty dxx^4H_1^{-1}\psi_1e^{-x}, \tag{37}$$

which will be needed later in the calculation of electron heat conductivity. Multiplying Eq. (25) by v^4 and performing the integration over all velocities one finds

$$J_Q = -3i\sqrt{\frac{\pi}{2}}\frac{\omega}{k^2 v_{Te}\lambda_{ei}}\left(1+\frac{5}{2}J_N-\frac{3}{2}J_T\right),$$
$$J_T = \frac{4\pi}{3n_0 v_{Te}^2}\int_0^\infty dv v^4\psi_0 F_0. \tag{38}$$

Here we have introduced the integral J_T, which accounts for the temperature perturbation $\delta T_e = e(J_T - J_N)\delta\phi$, and δT_e is defined as $\delta T_e = (4\pi m_e/3n_0)\int_0^\infty dv v^2(v^2 - 3v_{Te}^2)\delta f_0$.

The hydrodynamic solution of Eq. (22) corresponding to the classical heat transport [22,23] can be constructed from Eqs. (36) and (38), if we assume a two-term approximation, $\psi_0^{(1)} = (i\omega/kv_{Te})[c_0 L_0^{(1/2)} + c_1 L_1^{(1/2)}(x)]$, of the electron distribution and neglect the effect of high angular harmonics by setting $H_1 = 1$. Substitution of $\psi_0^{(1)}$ into the constraints (36) and (38) results in the following expressions:

$$c_0 = \frac{5}{2}c_1, \quad c_1 = i\frac{kv_{Te}}{4\omega(1+ir)}, \quad \text{where } r = \frac{32}{3\pi}\frac{kv_{Te}}{\omega}k\lambda_{ei}. \tag{39}$$

Note that $\psi_0^{(1)}$ does not have the proper behavior for large velocities, $v \gg v_{Te}$ and this is why the hydrodynamical approximation is valid only for very long-wavelengths. All necessary integrals can be written in terms of the parameter r,

$$J_N = -\frac{5}{8}\frac{1}{1+ir}, \quad J_R = -\frac{3}{8}\frac{1}{1+ir},$$
$$J_Q = 16\sqrt{\frac{2}{\pi}}\frac{1}{1+ir}. \tag{40}$$

On substitution of these expressions into (18) we have the well-known dispersion relation for hydrodynamic ion acoustic waves

$$\frac{\omega^2}{k^2 c_s^2} = 1 + \frac{2}{3+8ir}. \tag{41}$$

One can also see that the solution (39) corresponds to classical electron heat conductivity $\kappa = \kappa_0 = (128/3\pi)n_0 v_{Te}\lambda_{ei}$.

In order to describe the transition from the hydrodynamical limit into shorter wavelengths we have to account for more terms in the polynomial expansion of ψ_0. This transition occurs at $k\lambda_{ei} \lesssim Z^{-1/2}$ when the electron-electron collision term in Eq. (22) becomes comparable with the left-hand side. For any realistic ionization this transition occurs when $k\lambda_{ei}$ is larger than c_s/v_{Te} and therefore the parameter $r \gg 1$ and $|\psi_0| \ll 1$. These are the conditions of validity for Eq. (25), and so we can use the system (32) in order to find out where the deviation from hydrodynamics occurs. We assume that the parameter $Zk^2\lambda_{ei}^2 \ll 1$, neglecting the small imaginary part of the function $D(x)$ and the temperature gradient. Note that due to the conservation relations (A5) electron-electron collisions do not contribute to C_{0i} and C_{1i} in the limit $k\lambda_{ei} \ll 1$ and therefore at least the three-term expansion is needed. In that case electron-electron collisions contribute only to the matrix element $A_{22} \approx C_{22} = (3/4)\sqrt{\pi/2}/Zk\lambda_{ei}$ in Eq. (32). Therefore, the additional contribution to ψ_0 is proportional to $Zk\lambda_{ei}$ and in terms of coefficients c_n we find

$$c_0 = \frac{15\pi}{256}\frac{1}{k\lambda_{ei}}\left(1+\frac{2240}{9\pi\sqrt{2}}Zk^2\lambda_{ei}^2\right),$$
$$c_1 = \frac{3\pi}{128}\frac{1}{k\lambda_{ei}}\left(1+\frac{3200}{9\pi\sqrt{2}}Zk^2\lambda_{ei}^2\right), \tag{42}$$
$$c_2 = \frac{5\sqrt{2}}{3}Zk\lambda_{ei}.$$

The second terms in the parentheses in the expressions for c_0 and c_1 are the first corrections to the hydrodynamical solution Eq. (39). Large numerical coefficients appear in front of the expansion parameter. For example, the numerical coefficient in front of $Zk^2\lambda_{ei}^2$ in c_1 is approximately 80. Hence deviation from hydrodynamics already occurs for very small wave numbers, $k\lambda_{ei} \ll Z^{-1/2}$. However, the three polynomial expansion is not sufficient for the correct calculations of the coefficients c_i even in the long-wavelength limit. About 7 terms in the Laguerre expansion (28) should be kept in order to obtain accurate results for $k\lambda_{ei} \lesssim Z^{-1/2}$. Then we find

$$c_0 = \frac{15\pi}{256}\frac{1}{k\lambda_{ei}}\left(1+207Zk^2\lambda_{ei}^2\right),$$
$$c_1 = \frac{3\pi}{128}\frac{1}{k\lambda_{ei}}\left(1+264Zk^2\lambda_{ei}^2\right), \quad c_1 = 4.42Zk\lambda_{ei}. \tag{43}$$

The coefficient 264 in this expression for c_1 coincides with the analytical solution to the electron distribution function obtained in the limit $k\lambda_{ei} \ll Z^{-1/2}$ in Refs. [24] and [12] without the Laguerre polynomial expansion.

The electron velocity dependence of the distribution function ψ_0 is shown in Fig. 1 for $Z = 64$ and $k\lambda_{ei} = 0.01$. The exact ($N = 7$) solution to Eq. (32) is compared with the two and three polynomial approximations. The deviation from the hydrodynamic solution in the region of large velocities, $v \gtrsim (4-5)v_{Te}$, is obvious. It is responsible for the above-mentioned problem with the coefficients c_i. However, more important is the deviation by about 20% in the region of thermal electrons even for the expansion parameter $Zk^2\lambda_{ei}^2 \sim 10^{-2}$. This once more demonstrates the fact that the hydrodynamic approximation fails even for very small values of the collisionality parameter, $Z(k\lambda_{ei})^2$ and a more accurate solution to Eq. (25) is needed in the intermediate to collisionless region as was outlined in the previous section.

Figure 2 demonstrates the results of the numerical solution to Eq. (32) displayed in terms of the first anisotropic part of the electron distribution function δf_1 (A6) for different values of the collisionality parameter $k\lambda_{ei}$ and $Z = 64$. For the convenience of com-

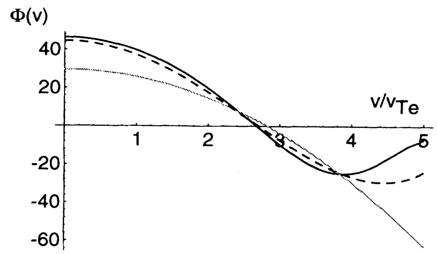

FIG. 1. The electron distribution function, $\psi_0 = i(\omega/kv_{Te})\Phi(v/v_{Te})$, in the region of small wave numbers which has been found from the analytical solution, Eq. (42) (dashed curve), two-term hydrodynamical approximation, Eq. (39) (gray curve), and the result of the numerical solution (solid curve) to Eq. (32) for $Z = 64$ and $k\lambda_{ei} = 0.01$ ($N = 7$). No temperature gradient.

parison to the classical limit we have normalized f_1 by the characteristic temperature gradient $-ik\lambda_{ei}\delta T_e/T_{e0} \equiv c_1 k\lambda_{ei}(\omega/kv_{Te})e\delta\phi/T_{e0}$. All functions δf_1 displayed in Fig. 2 possess a zero net current $\int_0^\infty dvv^3\delta f_1 = 0$ according to Eq. (36). In the limit of small $k\lambda_{ei} = 10^{-3}$ our anisotropic distribution function corresponds exactly to the classical solution (7) where the positive part of fast electrons carries the heat flux and the negative part represents the return current of slow electrons. Transition into smaller wavelengths affects δf_1 dramatically by increasing the number of slow electrons in δf_1 and therefore shifting all distributions towards small velocities. This is a consequence of the fact that the electron-electron collision term in Eq. (25) is proportional to v^{-3} and therefore has a larger effect on the slow electrons. In the limit of short wavelengths our solution agrees well with the weakly collisional asymptotics that is described in Sec. III C.

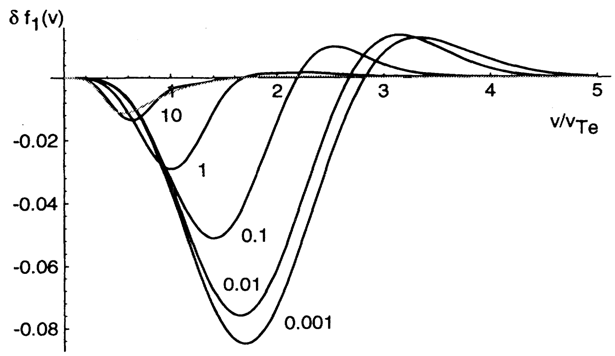

FIG. 2. The first asymmetric part of electron distribution function δf_1 for $Z = 64$, and $k\lambda_{ei} = 0.001$ ($N = 7$), $k\lambda_{ei} = 0.01$ ($N = 7$), $k\lambda_{ei} = 0.1$ ($N = 12$), $k\lambda_{ei} = 1$ ($N = 25$), and $k\lambda_{ei} = 10$ ($N = 50$). Gray line is the analytical solution of Eqs. (44) and (49) for $k\lambda_{ei} = 10$. The function δf_1 is normalized by $\lambda_{ei}/l_T = -ik\lambda_{ei}\delta T_e/T_{e0}$.

C. Weakly collisional limit

The Laguerre polynomial expansion (28) converges very slowly for $k\lambda_{ei} \gg 1$ and hence another approach for finding the electron distribution function in the short-wavelength region is required. As we have already discussed, the collisional term is small under condition (24) and therefore with the above assumptions an approximate solution can be found directly from Eq. (25). In the case of Maxwellian plasmas without temperature gradients we have $\psi_0 \approx \psi_{0T}$ where

$$\psi_{0T} = i\frac{\omega\nu_{ei}}{k^2 v_{Te}^2}\left[3\frac{v_{Te}^2}{v^2}H_1\left(\frac{kv}{\nu_{ei}}\right) - 1\right]. \quad (44)$$

This solution is well behaved for large velocities, but it diverges as v^{-5} in the region of small velocities. To resolve this divergence we either have to account for the frequency-dependent term on the left-hand side of Eq. (22) or for electron-electron collisions. The effect of the frequency dependence has already been discussed in Refs. [8,15], but we will see later that it is the electron-electron collisions that dominate in any practical case. The electron-electron collision contribution has recently been treated analytically by Maximov and Silin in Refs. [9–11] and in our paper [12]. We follow Ref. [12] in the derivation of the asymptotic solution. According to the collisional term given by Eqs. (26) and (27) in the region $v^2 \ll v_{Te}^2$ the last term in the right-hand side of Eq. (27) is much larger than the other terms and with an accuracy of about v^2/v_{Te}^2 we can write $G \approx (v^2/3\sqrt{2}v_{Te})d\psi_0/dv$. If we now substitute the collisionless solution $\psi_0 \propto v^{-5}$ into the reduced collisional term and compare it with the left-hand side of Eq. (25), we will find that it dominates when

$$(v/v_{Te})^7 Z(k\lambda_{ei})^2 \lesssim 1. \quad (45)$$

In this region of small velocities the kinetic equation (25) can be written as follows:

$$\frac{k^2v^2}{3\nu_{ei}(v)}\psi_0 = i\omega + \frac{1}{3Z}\sqrt{\frac{2}{\pi}}\nu_{ei}(v)\frac{v}{v_{Te}}\frac{d}{dv}\left(v^2\frac{d\psi_0}{dv}\right). \quad (46)$$

We have to find the regular solution to this equation which has the appropriate asymptotic behavior $\propto v^{-5}$ out of the region (45). We introduce the normalized variables $w = v/v_*$ and $Y = \psi_0/\psi_*$, where

$$v_* = v_{Te}\left(\frac{9\sqrt{\pi/2}}{Zk^2\lambda_{ei}^2}\right)^{1/7},$$

$$\psi_* = i\frac{\omega}{kv_{Te}}\left(9\sqrt{\frac{\pi}{2}}\right)^{2/7} Z^{5/7}(k\lambda_{ei})^{3/7} \quad (47)$$

represent the characteristic magnitudes of the velocity and the distribution function in the electron-electron collision-dominated region. The normalized equation reads

$$w^5 Y = 1 + \frac{d^2Y}{dw^2} + \frac{2}{w}\frac{dY}{dw}. \qquad (48)$$

In principle its solution can be written in terms of modified Bessel functions of the 1/7th order [9–11], but in fact it is more convenient to solve it numerically because all we need from (48) are the values of the function and its derivative at the origin and integrals (20) and (38). The numerical solution to this equation is shown in Fig. 3 together with the solution to the full equation (25) for the case of $Z = 64$ and $k\lambda_{ei} = 10$. The solution has the following asymptotics: $Y(w) \approx Y(0) - w^2/6$, where $Y(0) = c_\psi = 0.432$, for $w \ll 1$, and $Y(w) \approx 1/w^5$ for $w \gg 1$.

For the asymptotic evaluation of the integrals it is sufficient to use a simple analytical approximation for this function,

$$Y(w) \approx \frac{c_\psi}{1 + c_\psi w^5}, \qquad (49)$$

which is also shown in Fig. 3 for the symmetric part of the electron distribution function and in Fig. 2 for the asymmetric one. The accuracy of this approximation is better than 20% and is acceptable, because the corrections to Eq. (48) contribute to the solution of $Y(w)$ in the next order of the expansion parameters $\epsilon_1 = (v_*/v_{Te})^2 \approx (0.09Zk^2\lambda_{ei}^2)^{-2/7}$ and $\epsilon_2 = \left(9\sqrt{\pi/2}\right)^{-1/7} k\lambda_{ei}\epsilon_1^2$ which in practice are not that small because ϵ_1 contains a small fraction power. These two parameters equalize for $k\lambda_{ei} \sim 0.5Z^{2/3}$ which, as we see later, approximately corresponds to the wave number where collisional effects start to provide a significant contribution in comparison with electron Landau damping. Therefore, the correction to the electron distribution function related to $Y(w)$ is in fact interesting for smaller wave numbers and in this region $\epsilon_2 < \epsilon_1$. For $k\lambda_{ei} = 5$ and $Z = 10$ the parameter $\epsilon_1 = 0.4$ and this value, in fact, characterizes the accuracy of the analytical asymptotic analysis in the weakly collisional region. As we will see later this asymptotic solution allows us to obtain the qualitative

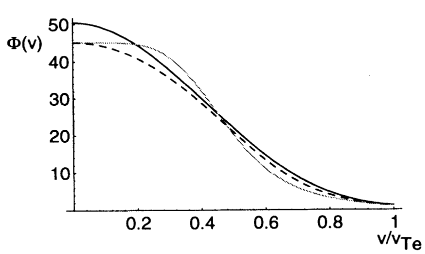

FIG. 3. The electron distribution function, $\psi_0 = i(\omega/kv_{Te})\Phi(v/v_{Te})$, in the region of small velocities, $v \lesssim v_{Te}$, which has been found from the numerical solution to Eq. (48) (dashed line), the analytical approximation (gray line), which has been represented by Eq. (49), and the result of the numerical integration (solid line) to Eq. (32) for $Z = 64$ and $k\lambda_{ei} = 10$ ($N = 50$). No temperature gradient.

scaling laws in the region of small wavelengths $k\lambda_{ei} \gg 1$, but a more accurate numerical solution to Eq. (25) is needed for quantitative agreement between this theory and Fokker-Planck simulations in the weakly collisional region, $k\lambda_{ei} \sim 1$.

IV. ELECTRON TRANSPORT COEFFICIENTS

A. Electron heat conductivity

The perturbation of the electron distribution function defined above allows us to investigate plasma transport properties. We discuss here the electron heat conductivity which has been the subject of many recent studies [8,13,16,17]. The diffusive electron heat flux associated with the ion acoustic wave $\delta q_e = \int d\mathbf{v}(m_e/2)v^2 v_z \delta f_e$ is related to the first harmonic of the electron distribution and therefore can be written in terms of the integral J_Q which was defined in Eqs. (37) and (38)

$$\delta q_e = -\frac{i}{3}\sqrt{\frac{2}{\pi}} n_0 T_{e0} v_{Te} k\lambda_{ei} J_Q \frac{e\delta\phi}{T_{e0}} \equiv -n_0 T_{e0}\frac{\omega}{k}\frac{e\delta\phi}{T_{e0}}\left(1 + \frac{5}{2}J_N - \frac{3}{2}J_T\right). \qquad (50)$$

Note that in the limit $k\lambda_{ei} \gg c_s/v_{Te}$ (23) the integrals J_N and J_T are much less than 1 and therefore using Eq. (18) one can find that $\delta q_e \approx -n_0 T_{e0} u_i$. Recall that the electron distribution function has been defined in the ion reference frame and by charge neutrality the hydrodynamic velocities of electrons and ions are equal. Therefore, the diffusive electron heat flux δq_e almost cancels the convective electron energy flux, $n_0 T_{e0} u_i$, resulting in an approximately zero net energy transport in the collisionless isothermal ion acoustic wave. However, in dealing with the electron heat conductivity we have to account for the diffusive part of the energy transport only.

The perturbation of electron temperature is related to the symmetric part of the distribution function $\delta T_e = e(J_T - J_N)\delta\phi$. Defining the electron heat conductivity κ related to ion acoustic waves as $\delta q_e/(-ik\delta T_e)$ we arrive at the expression

$$\kappa = \frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{J_Q}{J_T - J_N} n_0 v_{Te}\lambda_{ei}. \qquad (51)$$

It is convenient also to express κ in terms of the Laguerre expansion (28). Then using Eqs. (20) and (38) we see that the heat conductivity normalized by its classical limit $\kappa_0 = (128/3\pi)n_0 v_{Te}\lambda_{ei}$ of Braginskii and Spitzer and Härm [22,23] depends only on the coefficient c_1,

$$\frac{\kappa}{\kappa_0} = \frac{3\pi}{128 k\lambda_{ei} c_1}. \qquad (52)$$

From expressions (51) and (52) one can see that the reduction of the electron heat conductivity originates from the wavelength dependence of the temperature and density perturbations of the ion acoustic wave. Indeed, as we can see from Eq. (50), in the conditions of $k\lambda_{ei} \gg c_s/v_{Te}$

the ion acoustic wave is almost isothermal and, therefore, the electron heat flux does not depend on collisionality. This is why the numerator in Eq. (51) does not depend on the electron mean free path, λ_{ei}. In contrary, the magnitude of temperature perturbations depends significantly on the collisionality. Indeed, in the collisional region the ion acoustic wave is almost adiabatic and therefore the relative temperature disturbance is of the same order as the density and velocity disturbances, but the collisionless ion acoustic wave supports almost no temperature perturbations at all. This transition from the adiabatic to the isothermal limit manifests itself by the reduction of the electron heat conductivity. This is apparent in the calculation of the denominator in Eq. (51) in different regimes of particle collisionality.

In the limit of long-wavelengths we may use the analytical solution to the electron distribution function derived in Sec. III B. From Eqs. (43) and (52) one can see that the correction to the Braginskii heat conductivity is proportional to $264Zk^2\lambda_{ei}^2$ [12,24]:

$$\frac{\kappa}{\kappa_0} = \frac{1}{1+264Zk^2\lambda_{ei}^2}. \tag{53}$$

This expression certainly defines the characteristic wave numbers where the effect of flux inhibition occurs, $k\lambda_{ei} \sim 0.06Z^{-1/2}$, but it cannot be extended too far into the practically interesting region of smaller wave numbers.

We also can derive the expression for the electron heat conductivity in the weakly collisional case. Substitution of ψ_{0T} from (67) without the electron-electron collision correction into the integrals J_N and J_T shows a divergence at the lower limit of integration. The strongest divergence is exhibited by J_N, which is the integral related to the density perturbation. Therefore the electron-electron correction is particularly important for this integral. To evaluate the contribution from $v \lesssim v_*$ we represent ψ_{0T} as the sum of a low-velocity part $\psi_{0T}^{(L)} = \psi_* Y(v/v_*)$ and a high-velocity part $\psi_{0T}^{(H)} = \psi_{0T} - \psi_* Y(v/v_*)$. The part of $J_N = J_N^{(L)} + J_N^{(H)}$ related to the low-velocity component can be easily evaluated because this integral converges at $v \gg v_*$

$$J_N^{(L)} = \sqrt{\frac{2}{\pi}}\psi_* \frac{v_*^3}{v_{Te}^3}\int_0^\infty dw w^2 Y(w) = ic_\kappa \sqrt{\frac{\pi}{2}}\frac{\omega}{kv_{Te}} Z^{2/7}(k\lambda_{ei})^{-3/7}, \tag{54}$$

where $c_\kappa = 1.9$. The other part of J_N contains integrals that are already convergent at small velocities. They describe the effects of electron Landau damping and electron-ion collisions. In the collisional region $k\lambda_{ei} < 1$ this integral is of the order of $(k\lambda_{ei})^{-1}$, which is much smaller than $J_N^{(L)}$ in the region $k\lambda_{ei} \gg Z^{-1/2}$ (24) and therefore can be neglected. The contribution of $J_N^{(H)}$ in the short-wavelength region $k\lambda_{ei} > 1$ can be evaluated from the asymptotic expression for function $H_1(\xi) \approx \pi\xi/6$ for $\xi \gg 1$. Finally we have the expression $J_N^{(H)} = i\sqrt{\pi/2}(\omega/kv_{Te})$, which corresponds to electron Landau damping.

The thermal part of disturbance of the distribution function (44) in the short-wavelength limit $k\lambda_{ei} > 1$ gives the contribution $J_T^{(T)} = i(2/3)(\omega/kv_{Te})\sqrt{\pi/2}$ which is of the same order as $J_N^{(H)}$. There is also a logarithmic contribution of the order $(k\lambda_{ei})^{-1}\ln(v_{Te}/v_*)$, which comes from the region of small velocities $v \sim v_*$ but is small in comparison with $J_N(L)$ (54). We can neglect the contribution from J_T to the denominator in the collisional region $k\lambda_{ei} < 1$ for the same reasons. Therefore, in Maxwellian plasmas the denominator of Eq. (51) reads

$$J_T^{(T)} - J_N^{(T)} = -i\frac{\omega}{kv_{Te}}\sqrt{\frac{\pi}{2}}\left(\frac{1}{3} + c_\kappa Z^{2/7}(k\lambda_{ei})^{-3/7}\right). \tag{55}$$

The above described procedure of separating contributions from slow and thermal electrons overestimates the effect of electron-electron collisions because the real expansion parameter v_*/v_{Te} is not small enough in practice. This has already been noticed in Sec. III C. One can obtain a better approximation by numerically calculating the integrals J_N and J_T with the approximate distribution function $\psi_0(v) = c_\psi v^5 \psi_{0T}/(c_\psi v^5 + v_*^5)$ [12], which is an interpolation between Eqs. (44) and (49). This results in $c_\kappa \approx 1.2$ for $Z \gg 1$ and $k\lambda_{ei} \gg 1$. For large wave numbers we have the following analytical expression for the electron thermal conductivity:

$$\frac{\kappa}{\kappa_0} = \frac{9\sqrt{2\pi}}{128}\frac{1}{k\lambda_{ei} + 3c_\kappa Z^{2/7}(k\lambda_{ei})^{4/7}}. \tag{56}$$

In the limit of extremely large wave numbers this expression gives the exact collisionless limit $\kappa \propto 1/k$, which corresponds to the result of Ref. [3] and therefore accounts for the effect of the Landau damping. Note that the difference in the numerical coefficient from [3] is due to a different definition for the moments of the electron distribution function. Deviation from the collisionless limit happens already at large $k\lambda_{ei} \sim 20Z^{2/3}$. Starting from this value of k, the electron heat conductivity coefficient exhibits a much weaker dependence on wave number, $\kappa \propto (k\lambda_{ei})^{-4/7}$. This is due to the increase of the relative magnitude of the electron temperature perturbations. At the lower limit of applicability of Eq. (56) the heat conductivity is still ~ 30 times less than that in the hydrodynamical limit. Comparison of the analytical expression (56) with the results of numerical Fokker-Planck simulations [16] is shown in Fig. 4. Better quantitative agreement in the short-wavelength region can be achieved if we take the parameter $c_\kappa = 0.7$.

Equations (53) and (56) have been derived with the assumption of $Z \gg 1$ because the electron-electron collisions were neglected in all equations for the higher angular harmonics of the electron distribution function. It is possible to extend these equations into intermediate and small Z by following the prescription of Epperlein [16]. He proposed using the modified electron-ion mean free path $\lambda_{ei}^* = \lambda_{ei}\zeta(Z)/\zeta(\infty)$, where $\zeta(Z)/\zeta(\infty) = (Z+0.24)/(Z+4.2)$ is the ratio of charge-dependent coefficients in Braginskii's [22] electron thermal conductivity.

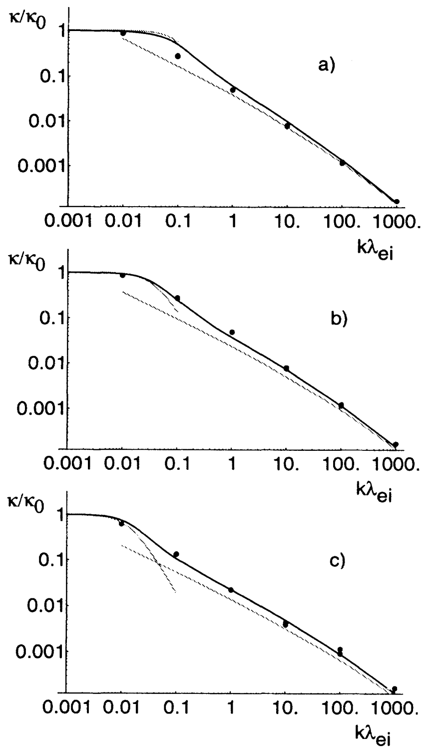

FIG. 4. The comparison of the interpolation analytical expression (57) for the electron heat conductivity coefficient in Maxwellian plasma (full lines) with the results of asymptotic expansions in the short- and long-wavelength limits, Eqs. (56) and (53) (gray lines). The numerical Fokker-Planck simulations from Ref. [16] and the numerical solutions to Eq. (32) are shown by dots for Z=1 (a), 8 (b), and 64 (c).

The asymptotic expressions (53) and (56) give insight into the construction of a numerical approximation for κ in the entire region of wavelengths. Namely, the coefficient c_κ in the short-wavelength asymptotics, Eq. (56), has to be substituted for a k-dependent function in such a way that it will smoothly describe the transition into the long-wavelength limit (53). Reasonably good agreement with the results of the numerical solution of the full kinetic equation (25) and with Fokker-Planck simulations [16] for Maxwellian plasmas can be achieved with the expression

$$\frac{\kappa}{\kappa_0} = \frac{9\sqrt{2\pi}}{128}\left(k\lambda_{ei}^* + \frac{3c_\kappa(1+264Zk^2\lambda_{ei}^{*2})}{128c_\kappa/3\sqrt{2\pi}+264(Zk^2\lambda_{ei}^{*2})^{5/7}}\right)^{-1}. \tag{57}$$

Figure 4 demonstrates the comparison between Fokker-Planck simulations and the analytical and numerical solutions to Eq. (25). The numerical solution to Eq. (25) coincides with the Fokker-Planck simulations to within an accuracy of a few percent, so it is hard to see any difference between the two sets of points in Fig. 4. A more accurate comparison between the Fokker-Planck simulations and the solution to Eq. (25) is demonstrated in Table I. The asymptotic analytical expressions (56) and (53) agree well with the numerics, but demonstrate a significant deviation in the intermediate region. The proposed interpolation, Eq. (57), with only one fitting coefficient $c_\kappa = 0.7$ gives the correct behavior for the electron heat conductivity in the entire region of wavelengths from the fully collisionless limit $k\lambda_{ei}^* \gg 20Z^{2/3}$ to the purely collisional region $k\lambda_{ei}^* \ll 0.1Z^{-1/2}$ and for different values of the ion charge ranging from $Z=1$ to $Z=64$. Notice the difference in numerical coefficients in Eq. (57) from our previous paper [12]. We found that the present set of coefficients produces the better agreement with numerical data. The strongest deviation of about 20% occurs for $k\lambda_{ei}^* \lesssim 0.1$ and $Z \sim 1$, which can still be considered as tolerable in many applications. This deviation could also be cured if the coefficient c_κ is allowed to be considered as a function of Z. It was also noticed in Ref. [16] that the heat conductivity in the intermediate region of electron collisionality has a negative imaginary part that increases for higher Z. This effect can also be described in our theory if we account for the small imaginary term in Eq. (22) or Eq. (30). We will return to this problem in Sec. IV C.

TABLE I. Comparison between the results of numerical solution to Eq. (25) (upper number in each cell) with the Fokker-Planck simulations in Ref. [16] (lower number) for the electron heat conductivity and ion acoustic damping in a Maxwellian plasma.

$k\lambda_{ei}^*$		κ/κ_0			γ/kc_s	
	$Z=1$	$Z=8$	$Z=64$	$Z=1$	$Z=8$	$Z=64$
0.01	0.933	0.849	0.615	0.0649	0.0714	0.0986
	0.96	0.849	0.616	0.0642	0.0709	0.0959
0.1	0.449	0.279	0.135	0.0139	0.0222	0.0456
	0.431	0.268	0.133	0.0144	0.0227	0.0451
1	0.0979	0.0493	0.0221	0.0098	0.0158	0.0310
	0.0955	0.0499	0.0224	0.0110	0.0156	0.0294
10	0.0134	0.00758	0.00394	0.0105	0.0139	0.0213
	0.0134	0.00808	0.00435	0.0109	0.0135	0.0192
100	0.00152	0.00124	0.000849	0.0107	0.0116	0.0138
	0.00157	0.00116	0.000927	0.0112	0.0120	0.0127

B. Ion acoustic wave damping

We first calculate the damping for the practically interesting wavelengths $k\lambda_{ei} > c_s/v_{Te}$ (23) where the function ψ_0 (44) and (67) is much smaller than 1 and is imaginary. The dispersion equation (21) can be solved as $\mathrm{Re}\omega = kc_s$, and the imaginary part of the ion acoustic frequency $\gamma = -\mathrm{Im}\omega$ is determined by the integrals J_R and J_N,

$$\frac{\gamma}{kc_s} = \frac{1}{2}\mathrm{Im}(J_N - J_R). \tag{58}$$

In terms of the Laguerre expansion coefficients we have the following expression for damping:

$$\frac{\gamma}{kc_s} = \frac{c_s}{v_{Te}\sqrt{\pi}} \sum_n \mathrm{Re}c_n \int_0^\infty dx\sqrt{x}e^{-x}L_n^{(1/2)}(x)S_T(x) + \frac{c_s}{2k\lambda_{ei}v_{Te}}\left(1 - \int_0^\infty \frac{dx}{H_1}e^{-x}\right) - \frac{2}{3\sqrt{\pi}}\frac{1}{kL_T}\int_0^\infty dx\frac{\sqrt{x}e^{-x}}{H_1}\left(6 - \frac{13}{2}x + x^2\right). \quad (59)$$

In the limit of short-wavelengths for the case of Maxwellian plasmas without temperature gradients substitution of ψ_{0T} from (67) without the electron-electron collisional term into the integrals (58) reveals a divergence at the lower limit of integration. The integral J_N exhibits the strongest divergence and we have already estimated in Eq. (54) due to the contribution from the small electron velocity region $v \lesssim v_*$. The integral J_R has only a logarithmic divergence at small velocities, which results in a term of the order $(k\lambda_{ei})^{-1}$. Following the same procedure as described above in Sec. IV A for the electron heat conductivity, we arrive at the following expression for ion acoustic damping in Maxwellian plasmas in the weakly collisional regime:

$$\frac{\gamma}{kc_s} = \sqrt{\frac{\pi}{8}}\frac{c_s}{v_{Te}}\left[1 + c_\kappa Z^{2/7}(k\lambda_{ei})^{-3/7}\right], \quad (60)$$

with $c_\kappa = 1.2$. The effect of electron-electron collisions is similar to the case of electron heat conductivity, but because the coefficient in front of c_κ is three times smaller it becomes important for smaller ion acoustic wave numbers $k\lambda_{ei} \lesssim c_\kappa^{7/3}Z^{2/3}$. The fact that electron-electron collisions can contribute significantly to ion acoustic damping in the limit $k\lambda_{ei} > 1$ has been recently recognized by Epperlein [16] who has found the dependence of $\gamma(k)$ on k by numerically solving the full Fokker-Planck equation. The analytical asymptotic expression has also been recently derived in Ref. [13] and independently in Ref. [12]. Expression (60) fits very well with the numerical results for different Z in the short-wavelength region $k\lambda_{ei} \gg 1$, but overestimates the electron-electron contribution as is demonstrated in Fig. 5. This discrepancy is due to the fact that the asymptotic expansion parameter $(v_*/v_{Te})^2$ is not small enough for the practical range of parameters. Much better quantitative agreement with Fokker-Planck simulations in the region $k\lambda_{ei} > 1$ can be obtained by using the numerical solution to the kinetic equation for ψ_0 (cf. Fig. 5 and Table I). Then the damping is expressed in terms of the Laguerre expansion coefficients by Eq. (59). The numerical solution can be modeled by using Eq. (60) with the numerical coefficient $c_\gamma \approx 0.5$.

Now turning to the long-wavelength limit, $k\lambda_{ei} < 1$ we see that there is no difference between the integrals J_R and J_T (20). Therefore, in this region the classical relation between ion acoustic wave damping and electron heat conductivity holds, $\gamma = n_0c_s^2/2\kappa$. Using this relation together with Eq. (53) we obtain the following expression for ion acoustic wave damping in the long-wavelength limit

$$\frac{\gamma}{kc_s} = \frac{3\pi}{256}\frac{c_s}{v_{Te}}\frac{1}{k\lambda_{ei}}(1 + 264Zk^2\lambda_{ei}^2). \quad (61)$$

The first term in parentheses corresponds to classical collisional damping in the limit of $k\lambda_{ei} > c_s/v_{Te}$ (23). The second accounts for the effects of electron-electron collisions due to a finite Z. We see from Eq. (61) that the deviation from hydrodynamical damping occurs for relatively small wave numbers $k\lambda_{ei} \sim 0.06Z^{-1/2}$. We can construct an approximate formula for ion acoustic wave damping which extends into the whole region of collisionalities if we combine Eqs. (60) and (61) in the following expression that has only one adjustable parameter:

$$\frac{\gamma}{kc_s} = \sqrt{\frac{\pi}{8}}\frac{c_s}{v_{Te}} \times \left(1 + \frac{c_\gamma}{k\lambda_{ei}^*}\frac{1 + 264Zk^2\lambda_{ei}^{*2}}{128c_\gamma/3\sqrt{2\pi} + 264(Zk^2\lambda_{ei}^{*2})^{5/7}}\right). \quad (62)$$

Here we have again introduced the modified electron mean free path as has been described in Sec. IV A in

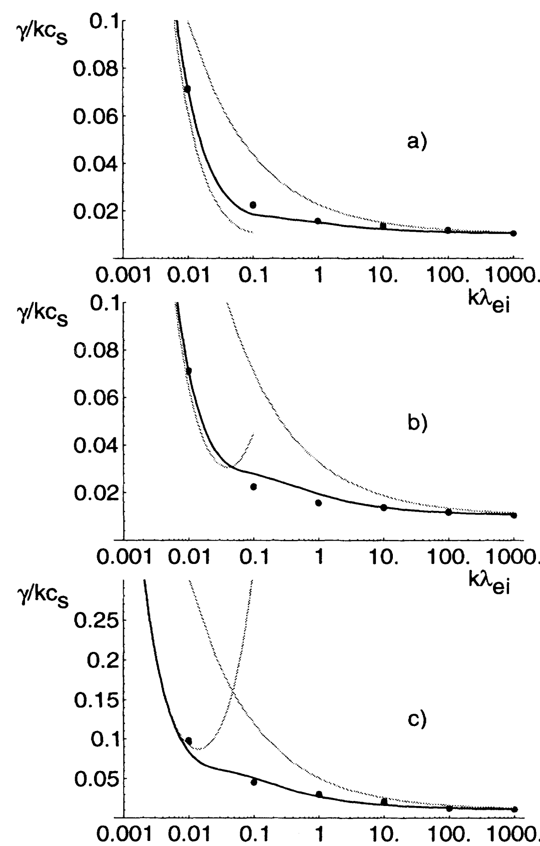

FIG. 5. The comparison of the analytical expression (62) for the ion acoustic wave damping in Maxwellian plasma (full lines) with the results of asymptotic expansions in the short- and long-wavelength limits, Eqs. (60) and (61) (gray lines). The numerical Fokker-Planck simulations from Ref. [16] and the numerical solutions to Eq. (25) are shown by dots for Z=1 (a), 8 (b), and 64 (c).

order to account for the effects of finite Z. The fitting parameter is taken to be $c_\gamma = 0.5$.

Comparison between this formula, numerical solution to Eq. (25), and Fokker-Planck simulations is shown in Fig. 5 and Table I. They demonstrate a good agreement for all parameters. In the figure there are no visible differences between the theory and simulations. The strongest deviation from numerical data is about 20% and it occurs for $k\lambda_{ei}^* \lesssim 1$ and $Z \gtrsim 1$, i.e., in the intermediate regime of collisionality and for low Z materials. This deviation could be also fixed if the coefficient c_γ is considered as a function of Z.

C. The limit of Lorentzian plasma

In spite of the fact that accounting for electron-electron collisions provides a fairly good description of ion acoustic damping and electron heat conductivity for practically interesting plasma parameters, it is instructive to consider the case of very high Z as well. It allows us to establish the relation between our theory and previous papers [8,15] where the electron-electron collisions have been neglected. Equations (46) and (48) show that the main function of electron-electron collisions in the short-wavelength region is to remove the divergence of the electron distribution function in the region of small velocities, which is important for the calculations of density and temperature disturbances. The small imaginary term proportional to $i\omega$ in the left-hand side of Eqs. (22) and (25) produces the same effect. It can be more important than electron-electron collision integral, if Z is large enough

$$Z > \left(\frac{v_{Te}}{c_s}\right)^{7/5} (k\lambda_{ei})^{-3/5}. \quad (63)$$

Because the right-hand side of this inequality contains the mass ratio, it can be satisfied only for very large Z (> 100). In that case we can neglect the electron-electron collision integral in Eq. (22) and write the explicit solution for ψ_0. Similarly to the collisional case, the only important contribution from the frequency-dependent term in the left-hand side of Eq. (22) comes into the electron density perturbation J_N (20) from the region of small velocities $v \sim v_{Te}(\omega/k^2 v_{Te}\lambda_{ei})^{1/5}$. Correspondingly, the contribution into J_N from this region will scale as

$$J_N \propto \frac{\omega}{k v_{Te}} \left(\frac{1}{k\lambda_{ei}}\right)^{3/5} \left(\frac{v_{Te}}{c_s}\right)^{2/5}. \quad (64)$$

In order to find the numerical coefficient in the front of this k dependence we performed the numerical integration of J_N and J_T (20) using the expression for ψ_0 from Eq. (22) without electron-electron collisions. Then, we find the following expression for the electron heat conductivity in Lorentzian plasmas:

$$\frac{|\kappa|}{\kappa_0} = \frac{9\sqrt{2\pi}}{128} \frac{1}{k\lambda_{ei} + 4.2(v_{Te}/c_s)^{2/5}(k\lambda_{ei})^{2/5}}. \quad (65)$$

This expression does not depend on Z and corresponds well to the results of papers [8,16] in the short-wavelength limit $k\lambda_{ei} \gtrsim 1$. Note that the electron heat conductivity is a complex quantity in that case and its phase is significant in the region of wave numbers where the nonlocal term in Eq. (65) dominates. The phase of κ decreases in collisionless, $k\lambda_{ei} > (v_{Te}/c_s)^{2/3}$, and strongly collisional, $k\lambda_{ei} < c_s/v_{Te}$, limits. The solution to Eq. (22) reproduces well the heat conductivity phase shift found in Ref. [16]. For example, the numerical solution to Eq. (22) gives $\arg\kappa = -35.2^\circ$ for $k\lambda_{ei} = 10$ and $Z \to \infty$, whereas simulations give $\arg\kappa = -31.9^\circ$. Similar agreement holds also for other parameters, if $k\lambda_{ei} \lesssim 0.1$.

D. The ion contribution to the ion acoustic damping

In the above analysis we have only considered the electron contribution to ion acoustic damping, with the assumption of cold ions. The effect of ion dissipation in the weakly collisional region has been described in Ref. [7]. The approximate Grad 21-moment solution to the ion kinetic equation results in the following expression for the ion contribution to the damping

$$\frac{\gamma_i}{k v_{Ti}} = k v_{Ti} \frac{\nu_i(1.49\nu_i^2 + 0.80k^2c_s^2)}{k^4c_s^4 + 4.05\nu_i^2k^2c_s^2 + 2.33\nu_i^4} + \frac{7(k\lambda_i)^2}{1 + 7(k\lambda_i)^2} L_i, \quad (66)$$

where $\nu_i = 4\sqrt{\pi}Z^4e^4n_i\Lambda_i/3\sqrt{m_i}T_i^{3/2}$ is the ion-ion collision frequency, Λ_i is the ion Coulomb logarithm, v_{Ti} is the ion thermal velocity, $\lambda_i = v_{Ti}/\nu_i$ is the ion mean free path, and $L_i = \sqrt{\pi/8}(ZT_e/T_i)^{3/2}(3 + ZT_e/T_i)^{1/2}\exp[-(3 + ZT_e/T_i)/2]$ is the ion Landau damping contribution. Note that the second term on the right-hand side does not result from the 21-moment method and is added phenomenologically to account for ion Landau damping.

The relative ion contribution depends on the ratio $\lambda_{ei}/\lambda_i = Z^2T_e/T_i\sqrt{2}$. Usually this parameter is large and hence the ion contribution is important in the limit of short wavelengths. One can see from Eq. (66) that the ion contribution to the ion acoustic wave damping has a maximum at $k\lambda_{ei} \sim (ZT_e/T_i)^{3/2}$ and then decreases as k^2 in the ion collisional region. In Fig. 6 we show the wavelength dependence of the total ion acoustic damping for different electron to ion temperature ratios. The gray line demonstrates the electron contribution; it corresponds to cold ions, $T_e/T_i \to \infty$. The ion contribution increases in the region $k\lambda_{ei} > 1$ as the ion temperature increases, but there is no significant ion contribution in the long-wavelength region, $k\lambda_{ei} < 1$, where electrons dominate.

V. RETURN CURRENT INSTABILITY

Now we turn our attention to the case of plasmas with temperature gradients and discuss the gradient contri-

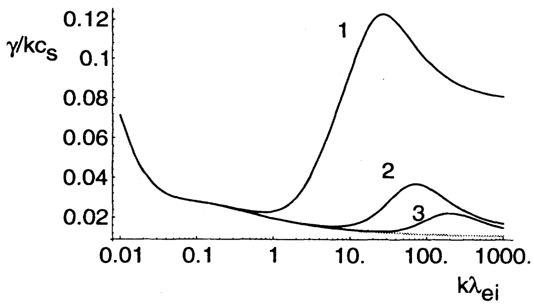

FIG. 6. The wavelength dependence of the ion (66) and electron (62) parts of the ion acoustic damping (full lines) and only the electron part of ion acoustic damping (light line) for the plasma with $Z = 8$ and different electron-to-ion temperature ratios $T_e/T_i = 1$ (1), 2 (2), and 4 (3).

bution to the perturbation of the electron distribution function. Because the equation for δf_e is linear, the temperature gradient produces an additive contribution. In the short-wavelength limit $k\lambda_{ei} \gg 1$ we may neglect the electron-electron collision term in Eq. (22) and find the following expression for the gradient-dependent part of the distribution function

$$\psi_{0\nabla} = -i\frac{\nu_{ei}}{kv}\left[3\frac{v_{Te}^2}{v^2}H_1\left(\frac{kv}{\nu_{ei}}\right) - 1 + v_{Te}^2\frac{d}{dv}\frac{1}{v}\right]\psi_\nabla. \quad (67)$$

This can be used for the straightforward calculation of ion acoustic wave damping, because all integrals in Eq. (58) are convergent. Hence, electron-electron collisions do not have a considerable effect on this part of the distribution function under the conditions (24). The calculation of the gradient-dependent parts of integrals J_N and J_R gives us the following expression for the ion acoustic wave increment

$$\frac{\gamma_\nabla}{kc_s} = -\frac{1}{2\sqrt{\pi}kL_T}\int_0^\infty \frac{dx}{\sqrt{x}}e^{-x}(4-x)(3H_1 + 2 - 2x). \quad (68)$$

Here the term that is proportional to H_1 on the right-hand side of Eq. (68) dominates. In that case γ_∇ asymptotically approaches its collisionless value $-(3\sqrt{2\pi}/8)(\lambda_{ei}/L_T)kc_s$.

The negative value of γ_∇ means that the temperature gradient decreases the total ion acoustic damping rate (58) and can ultimately change its sign. This case corresponds to the instability of ion acoustic waves. In the limit of large wave numbers where all collisional effects are insignificant one can compare γ_∇ with the Landau damping term in Eq. (60) and arrive to the following instability criterion:

$$P \equiv \frac{3}{2}\frac{v_{Te}}{c_s}\frac{\lambda_{ei}}{L_T} > 1. \quad (69)$$

In terms of the electron heat flux this corresponds to the well-known threshold value of $q_{\rm th}$ [25,26]

$$q_{\rm th} = \zeta(Z)n_0T_{e0}v_{Te}\lambda_{ei}/L_{Tth} \approx 9.05n_0T_{e0}c_s. \quad (70)$$

Note that this expression for $q_{\rm th}$ is 5.4 times larger than that derived in the original paper of Forslund [6] and recently used in Ref. [7]. The cause of this discrepancy is in the difference of reference distribution functions. It was shown in Refs. [25,26] that the polynomial expansion used in Refs. [6,7] does not satisfy the collisional electron kinetic equation (1) for the reference state.

The kinetic approach allows us to investigate the return current instability in the region where collisions become important. With decreasing wave number the instability threshold is slightly increased by a larger ion acoustic damping (60) caused by electron-electron collisions. γ_∇ stays approximately the same as these collisions have an insignificant effect upon it. Therefore for $k\lambda_{ei} \sim 1$ the instability threshold is approximately $Z^{2/7}$ times larger than (69). In the long-wavelength region the threshold decreases again. This part corresponds to the hydrodynamical instability derived recently in Ref. [7]. An approximate analytical solution to the kinetic equation (25) in the long-wavelength region can be derived similarly to that in Sec. III B using the Laguerre polynomial expansion or exact integral solution similarly to [24].

$$\begin{aligned} c_0 &= \frac{15\pi}{256}\frac{1}{k\lambda_{ei}}\left[1 + 207Zk^2\lambda_{ei}^2(1-5.3P)\right], \\ c_1 &= \frac{3\pi}{128}\frac{1}{k\lambda_{ei}}\left[1 + 264Zk^2\lambda_{ei}^2(1-5.3P)\right], \quad (71) \\ c_2 &= 4.42Zk\lambda_{ei}(1-4.7P). \end{aligned}$$

Substitution of the coefficient c_1 into Eq. (59) gives us the following expression for ion acoustic wave damping in the long-wavelength limit [cf. Eq. (61)]:

$$\frac{\gamma}{kc_s} = \frac{3\pi}{256}\frac{c_s}{v_{Te}}\frac{1}{k\lambda_{ei}}[1 + 264Zk^2\lambda_{ei}^2(1-5.3P)]. \quad (72)$$

The first term in square brackets corresponds to the classical collisional damping. The second one accounts for the finite-Z effect due to electron-electron collisions. The coefficient in front of P is larger than 1 and, therefore, the temperature gradient may destabilize ion acoustic waves even for small P below the threshold of short-wavelength ion acoustic instability.

Figures 7(a) and 8(a) demonstrate the effect of temperature gradient on the ion acoustic wave damping found from the numerical solution to Eq. (25) for two different values of Z and $P = 1$. One can see that in the long-wavelength region, $k\lambda_{ei} < 1$, there is already a region of negative γ that corresponds to unstable ion acoustic waves. For given P the growth rate increases with Z and shifts towards longer wavelengths. The gradient-dependent part of the ion acoustic damping can be approximated with the following expression:

$$\frac{\gamma_\nabla}{kc_s} = -\frac{3}{4}\sqrt{\frac{\pi}{2}}\frac{\lambda_{ei}}{L_T}\left[\left(1 + \frac{c_\nabla}{\sqrt{k\lambda_{ei}}}\right)^{-1} + \frac{1}{80Zk\lambda_{ei}}\right]^{-1}, \quad (73)$$

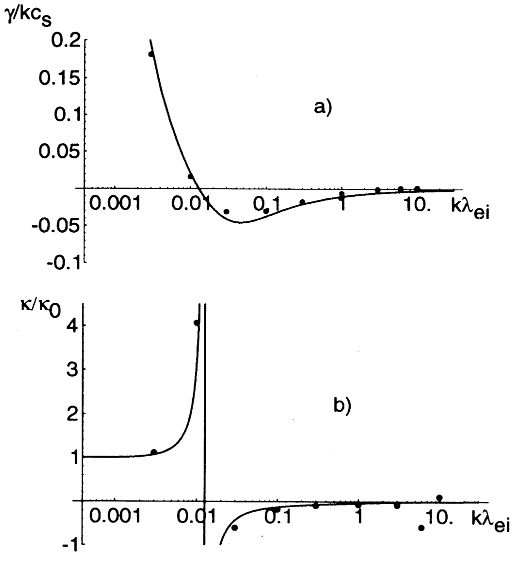

FIG. 7. The wave-number dependence of the ion acoustic wave damping (a) and the normalized electron heat conductivity coefficient (b) in a plasma with the temperature gradient, $P = 1$ and $Z = 8$. Dots correspond to the numerical solution to Eq. (32), solid curves are the analytical approximations (73) and (75).

which coincides well with the numerical solution to Eq. (32). The coefficient c_∇ slightly changes with Z from $c_\nabla = 1.9$ for $Z = 8$ to $c_\nabla = 2.2$ for $Z = 64$. It can be seen from Eq. (73) that the threshold for the long-wavelength instability corresponds to $P \approx 0.4$. This is 2.5 times

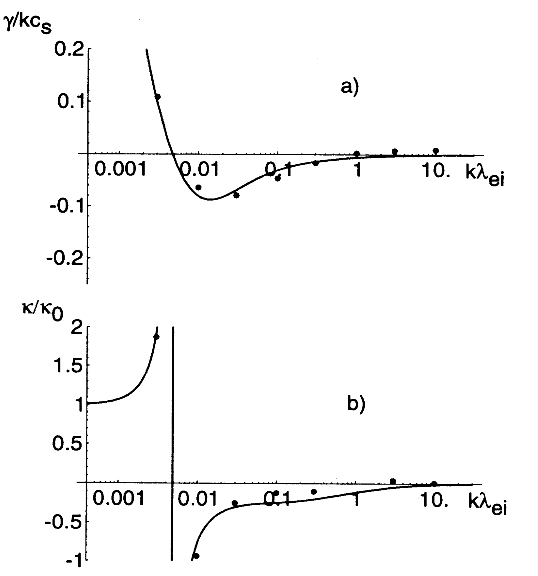

FIG. 8. The wave-number dependence of the ion acoustic wave damping (a) and the normalized electron heat conductivity coefficient (b) in a plasma with the temperature gradient, $P = 1$ and $Z = 64$. Dots correspond to the numerical solution to Eq. (32), solid curves are the analytical approximations (73) and (75).

lower than the known short-wavelength threshold (69). Note that this long-wavelength part of the ion acoustic instability agrees with the results of our previous paper [7] where a phenomenological expression for the nonlocal electron heat conductivity was used.

It can be seen from these figures that the instability onset is accompanied by a negative electron heat conductivity. This fact follows also from analytical consideration. In the short-wavelength region, $k\lambda_{ei} \gg 1$, we have an additional contribution to the denominator of Eq. (51) related to the temperature gradient

$$(J_T - J_N)_\nabla = \frac{i}{\sqrt{\pi} k L_T} \int_0^\infty \frac{dx}{\sqrt{x}} e^{-x}(4-x)[(3-2x)H_1 + 2].$$

It has the opposite sign to the Maxwellian contribution (55) and can therefore change the sign of κ

$$\frac{\kappa}{\kappa_0} = \frac{9\sqrt{2\pi}}{128} \frac{1}{k\lambda_{ei}(1 - 8P/\pi) + 3c_\kappa Z^{2/7}(k\lambda_{ei})^{4/7}}. \quad (74)$$

Comparing this expression with the instability threshold (69) one can see that κ as a function of P for given $k\lambda_{ei} \gg 1$ goes to infinity and then changes sign even before the instability threshold has been reached. Combining the long-wavelength asymptotics (71) with Eq. (74) we find the following interpolation for $\kappa(k)$ in a plasma with a temperature gradient

$$\frac{\kappa}{\kappa_0} = \frac{9\sqrt{2\pi}}{128} \left[k\lambda_{ei} + \frac{3c_\kappa(1 + 264 Z k^2 \lambda_{ei}^2)}{128 c_\kappa / 3\sqrt{2\pi} + 264 Z^{5/7}(k\lambda_{ei})^{10/7}} - \frac{8}{\pi} P \left(\frac{1}{k\lambda_{ei} + 2.7\sqrt{k\lambda_{ei}}} + \frac{1}{80 Z k^2 \lambda_{ei}^2} \right)^{-1} \right]^{-1}. \quad (75)$$

This formula together with (73) is shown in Figs. 7(b) and 8(b). They agree reasonably well with exact solution to Eq. (32).

Obviously, a negative heat conductivity illustrates the inapplicability of thermodynamical definitions to a nonequilibrium plasma, but it also gives us another physical explanation for the heat flux driven instability that has been already proposed in Ref. [7]. First of all we note that the temperature gradient does not contribute to the electron heat flux associated with ion acoustic wave, which remains unchanged (cf. Sec. IV A) and is independent of wavelength. The sign change in the denominator of Eq. (51) manifests itself in the change of the temperature perturbation phase with respect to the heat flux perturbation. Therefore, heat flux can enhance the temperature perturbations in contrast to the usual situation in equilibrium plasmas where the heat flux dissipates the temperature disturbances. This enhancement ultimately results in the instability, if the external heat flux is large enough.

Figure 9 demonstrates the excitation of the ion acoustic wave instability for different temperature gradients when the ion part of the ion acoustic damping is also taken into account according to Eq. (66). The growth rate $\gamma \approx 0.07kc_s$ can be easily achieved for $k\lambda_{ei} = 0.2$ and a temperature gradient $L_T \approx 45\lambda_{ei}$ ($P = 2$). For a

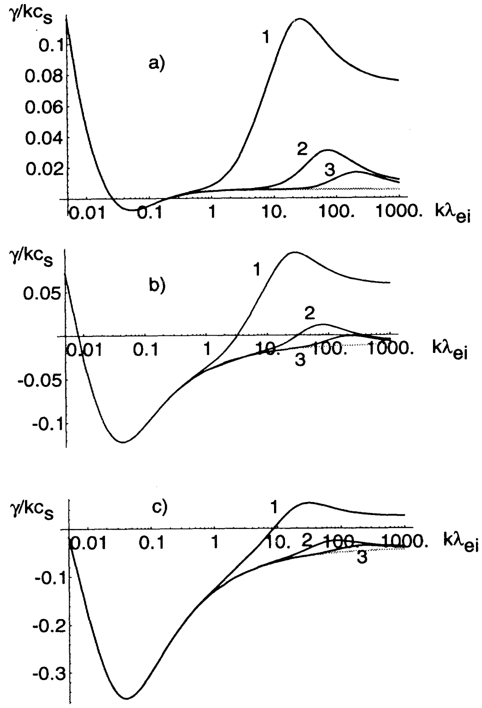

FIG. 9. The wavelength dependence of the total ion acoustic damping (full lines) and only the electron part of ion acoustic damping (light lines) for the plasma with $Z = 8$, electron to ion temperature ratio $T_e/T_i = 1$ (1), 2 (2), and 4 (3), and different electron temperature gradient $P = 0.5$ (a), 2 (b), 5 (c).

laser-produced plasma with an electron density $n_0 \sim 10^{21}$ cm^{-3} and temperature 1 keV the instability growth time $1/|\gamma| \sim 100$ ps. This long-wavelength instability corresponds to the excitation of relatively large-scale perturbations, $1/k \sim 5$ μm, and therefore their development should be considered in the context of the global hydrodynamic evolution of the plasma. The scale of these perturbations is already comparable with the temperature scale length and a more accurate analysis of the instability is needed for any particular plasma conditions. The short-wavelength ion acoustic instability is less dependent on plasma geometry because $kL_T \gg 1$. It has a larger threshold but it also has a larger growth rate. The short-wavelength ion acoustic waves may be responsible for the nonthermal scattering of electromagnetic waves and anomalous transport coefficients in the time scale of $\gtrsim 10$ ps as has been discussed elsewhere [26,27].

VI. CONCLUSIONS

In this paper we have discussed analytically the effects of electron-electron collisions on ion acoustic wave damping and provided a fully kinetic treatment of the heat flux driven instability for arbitrary electron collisionality. We can summarize our results in the following way.

We have found that the deviation of the ion acoustic damping and electron heat conductivity from well-known classical collisional expressions appears as early as at $k\lambda_{ei} \sim 0.06Z^{-1/2}$, and electron-electron collisions dominate in a wide parameter range up to $k\lambda_{ei} \sim (2-20)Z^{2/3}$. Our analysis shows that electron-electron collisions have a strong effect on the electron distribution and therefore their neglect in Refs. [8,15] cannot be justified. The k dependence of ion acoustic wave damping and electron heat conductivity agrees well with the results of kinetic Fokker-Planck simulations [16] of ion acoustic waves. Note that the scaling of the electron heat conductivity, $\kappa \propto k^{-4/7}$, associated with ion acoustic waves in the weakly collisional region, $k\lambda_{ei} \sim 1$ differs significantly from the case of inverse bremsstrahlung absorption [4,9,10] where the law $\kappa \propto k^{-10/7}$ has been found. In both cases the effect of electron heat conductivity inhibition originates from the change in the electron distribution function in the region of small electron velocities, $v \lesssim v_{Te}$, and therefore manifests itself through the anomalous k dependence of density and temperature perturbations. The direct effect of electron-electron collisions on the electron heat flux is much weaker. This fact has been explicitly demonstrated here where we have discussed the effect of the temperature gradient on the electron heat conductivity.

The return current instability has the lowest threshold in the long-wavelength region, $k\lambda_{ei} < 1$, where it is found to be more than two times lower than that previously known for short-wavelength ion acoustic waves. This long-wavelength instability may be manifested in strongly inhomogeneous plasma flows like jets. For typical laser plasma parameters these structures may develop in the nanosecond time scale. The instability of the electron heat flux may also affect the electron heat transport and overall plasma hydrodynamics due to the additional scattering of electrons from the ion acoustic waves. However, we defer these problems for future publications.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE MATRIX ELEMENTS OF THE COLLISION OPERATOR

Straightforward integration of Eq. (29) with the $L_m^{(1/2)}(x)$ gives the following expression for the contribution from the electron-electron collision term

$$C_{mn} = -\frac{3}{Zk\lambda_{ei}} \int_0^\infty dx L_m^{(1/2)}(x) \frac{d}{dx}[e^{-x} G_n(x)], \quad \text{(A1)}$$

where

$$G_n(x) = \int_0^x dy \left(\frac{d^2}{dy^2}L_n^{(1/2)}(y)\right)\left[\gamma\left(\frac{3}{2},y\right) - \frac{2}{3}y^{3/2}e^{-y}\right] - \frac{2}{3}x^{3/2}\int_x^\infty dy \left(\frac{d^2}{dy^2}L_n^{(1/2)}(y)\right)e^{-y}. \tag{A2}$$

Integrating Eq. (A1) by parts, using $G(0) = 0$, and then integrating each term in Eq. (A2) by parts leads to

$$C_{mn} = \frac{3}{Zk\lambda_{ei}}\int_0^\infty dx e^{-x}\left(\frac{d}{dx}L_m^{(1/2)}(x)\right)\left[\left(\frac{d}{dx}L_n^{(1/2)}(x)\right)\gamma\left(\frac{3}{2},x\right)\right.$$
$$\left. -\frac{2}{3}\int_0^x dy\left(\frac{d}{dy}L_n^{(1/2)}(y)\right)y^{3/2}e^{-y} - \frac{3}{2}x^{3/2}\int_x^\infty dy\left(\frac{d}{dy}L_n^{(1/2)}(y)\right)e^{-y}\right]. \tag{A3}$$

The above expression can now be simplified by making use of some of the properties of the generalized Laguerre polynomials. The derivatives of the Laguerre polynomials can be rewritten by noting the relation $(d/dx)L_n^{(1/2)}(x) = -L_{n-1}^{(3/2)}(x)$. The last term on the right-hand side of Eq. (A3) can be simplified by changing the order of the integration over x and y so that it looks like the second term but with $m \rightleftharpoons n$. The double integral can then be converted into a one-dimensional integral since we have the following relation due to the Rodrigues' formula for the generalized Laguerre polynomials,

$$\int_0^x dy L_m^{(1/2)}(y)y^{3/2}e^{-y} = \frac{1}{m-1}L_{m-2}^{(5/2)}(x)x^{5/2}e^{-x}. \tag{A4}$$

This gives the final result for C_{mn} shown in Eq. (34).

From Eq. (34) it can be seen that the collision operator is symmetric, $C_{mn} = C_{nm}$. Furthermore we have the following invariants of the collision operator:

$$C_{n0} = C_{0n} = 0; \quad C_{1n} = C_{n1} = 0. \tag{A5}$$

These can be easily seen from Eqs. (A1) and (A2) since the derivatives vanish.

All the useful quantities can now be written in terms of the expansion coefficients $c_n = A_{mn}^{-1}b_m$. For example, the first asymmetric part of the distribution function $\delta f_1(x)$ (15) is given by

$$\delta f_1(x) = -i\frac{4}{3}\sqrt{\frac{2}{\pi}}\frac{k\lambda_{ei}x^2}{H_1}\psi_1(x)\frac{e\delta\phi}{T_{e0}}F_0, \tag{A6}$$

where according to (19)

$$\psi_1(x) = i\frac{\omega}{kv_{Te}}\sum_n c_n L_n^{(1/2)}(x) - i\frac{\omega}{kv_{Te}}\frac{3\sqrt{\pi}}{4}\frac{H_1-1}{k\lambda_{ei}x^{3/2}}$$
$$-i\frac{\lambda_{ei}}{L_T}\frac{H_1-1}{k\lambda_{ei}x}\left(6-\frac{13}{2}x+x^2\right). \tag{A7}$$

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